1. Proof of Proposition 4

Part (i). As $F'$ is non-monotonic and continuously differentiable, there exists $w^{**}$ such that $F''(w^{**}) = 0$. We know that $\theta(w)$ and $x[s(w)]$ are strictly decreasing in $w$ for all $w \geq w^*$. Therefore, $F'(w)$ is also strictly decreasing in $w$ for $w \geq w^*$. Taking the second derivative of $F$ yields

$$\frac{F''(w)}{F'(w)} = \frac{\theta'(w)}{\theta(w)} + \frac{x'[s(w)]s'(w)}{x[s(w)]}$$  \hspace{1cm} (1)

During the proof of Proposition 2, we show that

$$\frac{x'[s(w)]}{x''[s(w)]} = \frac{\phi(w)}{\alpha(\theta(w))(y - w)(w - R)}$$  \hspace{1cm} (2)

where $\phi$ is defined in the proof of Proposition 2. Using (2) and (3), we have:

$$\frac{F''(w)}{F'(w)} = \frac{\psi(w)}{\alpha(\theta(w))(y - w)(w - R)}$$  \hspace{1cm} (4)

where $\psi$ is defined in the proof of Proposition 3. In turn,

$$\psi'(w) = \gamma[s(w)]\phi'(w) - 1 + s'(w)\gamma'[s(w)]\phi(w)$$  \hspace{1cm} (5)

As $\phi'(w) \leq 0$ for all $w \leq w^*$, we obtain

$$\psi'(w) \leq -1 + s'(w)\gamma'[s(w)]\phi(w) \text{ for all } w \in [R, w^*]$$  \hspace{1cm} (6)

The properties of the matching technology together with equation (3) yield

$$\psi'(w) \leq -1 + \frac{\gamma[s(w)]\phi(w)^2}{\alpha[\theta(w)](y - w)(w - R)} \text{ for all } w \in [R, w^*]$$  \hspace{1cm} (7)

As $\gamma[s(w^{**})]\phi(w^{**}) = w^{**} - R$, we finally obtain

$$\psi'(w^{**}) \leq -1 - \frac{\alpha[\theta(w^{**})]}{\gamma[s(w^{**})]} w^{**} - R \leq 0$$  \hspace{1cm} (8)
Consequently, the equation $\psi(w) = 0$ has a unique root $w^* \in (R, w^*)$.

Part (ii). Let $\chi(w) = \gamma [s(w)] \phi(w) - [1 - \alpha \theta(w)] (w - R)$. As $G'$ is non-monotonic and continuously differentiable, there exists $w^{**} \in (R, y)$ such that $G''(w^{**}) = 0$. As $F'(w)$ is strictly increasing for all $w \in [R, w^{**}]$, $G'(w)$ is also strictly increasing on $[R, w^{**}]$. Computing $G''$ and eliminating $\theta'(w)/\theta(w)$ by means of (2) gives:

\[
\frac{G''(w)}{G'(w)} = \frac{F''(w)}{F'(w)} - \alpha \theta(w) \frac{\theta'(w)}{\theta(w)} = \frac{F''(w)}{F'(w)} + \frac{1}{y - w} \tag{9}
\]

Using relation (4) that defines $F''(w)$, we obtain:

\[
\frac{G''(w)}{G'(w)} = \frac{\chi(w)}{\alpha \theta(w)(y-w)(w-R)} \tag{10}
\]

As $\phi(w) \leq 0$ for all $w \geq w^*$, we have $\chi(w) \leq 0$ for all $w \geq w^*$. Taking the derivative of $\chi$ gives

\[
\chi'(w) = \gamma [s(w)] \phi'(w) + s'(w) \gamma' [s(w)] \phi(w) + \theta'(w) \alpha' \theta(w) [w-R] - (1 - \alpha \theta(w)) \tag{11}
\]

During the proof of Proposition 2, we show that

\[
\alpha' \theta > -\frac{\alpha \theta}{\theta} (1 - \alpha \theta) \tag{12}
\]

\[
\phi'(w) < \frac{1 - \alpha \theta(w)}{y - w} (y - R) - 1 = -\frac{\phi(w)}{y - w} \tag{13}
\]

Using (12), (13), the properties of the matching technology, and the definition of $\chi$, we get

\[
\chi'(w) \leq \frac{\gamma [s(w)] \phi(w)^2}{\alpha \theta(w) (y-w)(w-R)} - (1 - \alpha \theta(w)) - \frac{\chi(w)}{y-w} \text{ for all } w \leq w^* \tag{14}
\]

As $\gamma [s(w^{**})] \phi(w^{**}) = (1 - \alpha \theta(w^{**})) (w^{**} - R)$, we finally obtain

\[
\chi'(w^{**}) \leq -1 - \frac{1 - \alpha \theta(w^{**})}{\alpha \theta(w^{**})} \frac{w^{**} - R}{y - w^{**}} < 0 \tag{15}
\]

Consequently, the equation $\chi(w) = 0$ has a unique root $w^{**} \in [w^*, w^*]$.

2. Proof of Proposition 5

We have

\[
\frac{G''(w)}{G'(w)} = \frac{\chi(w)}{\alpha \theta(w)(y-w)(w-R)} \tag{16}
\]

Using the definition of $\chi(w)$ given in the proof of Proposition 4, equation (16) becomes:

\[
\frac{G''(w)}{G'(w)} = \frac{\gamma(y-R)}{(y-w)(w-R)} - \frac{1 + \gamma - \alpha}{\alpha(y-w)} \tag{17}
\]
Integrating this equation with the condition \( \int_R^y G'(w)dw = 1 \) yields

\[
G'(w) = \left( \frac{w - y}{\xi - y} \right)^{\frac{1}{\alpha}} \left( \frac{w - R}{\xi - R} \right)^{\gamma} \frac{1}{\alpha - \gamma} \frac{1}{\alpha - \gamma} \frac{1}{(\xi - y) \alpha} \frac{(w - R) \gamma}{\xi - R} d\xi
\]  

(18)

The cdf of the actual normalized wage is such that

\[
H_G(\omega) = \Pr \left( \frac{w - R}{y - R} \leq \omega \right) = \Pr (w \leq R + \omega(y - R)) = G[R + \omega(y - R)]
\]

Therefore one gets \( H'_G(\omega) = (y - R)G'[R + \omega(y - R)] \), and the result follows.

3. Proof that there are no multiple offers

We closely follow Mortensen (1986) who shows in the standard job-search model that the probability of receiving more than one offer conditional on the fact that the worker receives at least one offer tends to 0 as the time interval tends to 0. The proof must be adapted to account for the fact that there are a continuum of markets in our model. Mortensen discretizes time, and interprets continuous time as a case in which the time interval between two dates tends to 0. We not only discretize time but also space: the wage distribution is cut into intervals of equal distance. The case of a continuum of markets corresponds to the case where such distance tends 0.

Consider the function \( (w) = x[s(w)] \theta(w) m[\theta(w)] \) defined over \([R, y]\). Our proof does not depend on the particular form taken by the function \( \lambda(w) \) in our model. It is valid for any function \( \lambda(w) \) that is positive and continuous on the interval \([R, y]\). Cut the interval \([R, y]\) into \( n \) intervals of the same length \( dw = (y - R)/n \). On interval \( i \in \{1, ..., n\} \), there is a unique wage \( w_i = R + (i - 1) dw \). Now, consider interval \( i \), and cut it into \( m \) intervals. Assume that the probability of receiving an offer from any such interval is \( \lambda_i dw/dt/m \) over the period \( dt \), with \( \lambda_i = \lambda(w_i) \).

Let \( X_i \) be the number of offers received from interval \( i \) over the period \( dt \). The probability of receiving \( k_i \in \{0, ..., m\} \) offers is:

\[
\Pr (X = k_i) = C^k_{m} (\lambda_i dw/dt/m)^{k_i} (1 - \lambda_i dw/dt/m)^{m-k_i}
\]

(20)

As \( m \to \infty \), it tends to

\[
\Pr (X = k_i) = e^{-\lambda_i dw/dt} \frac{(\lambda_i dw/dt)^{k_i}}{k_i!}
\]

(21)

Hence, \( X_i \) follows the Poisson law of parameter \( \lambda_i dw/dt \).

Now, consider the random variable \( X = \sum_{i=1}^{m} X_i \) which is the total number of offers received from all the intervals over the period \( dt \). As the different variables are independent draws from Poisson laws, the sum of the draws also

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follows a Poisson law, whose parameter is the sum of the parameters of the different Poisson laws. Hence,

\[ \Pr (X = k) = \Pr \left( \sum_{i=1}^{n} X_i = k \right) = e^{-\lambda dt} \frac{\lambda^k}{k!} \], \quad \text{with} \ \lambda = \sum_{i=1}^{n} \lambda_i dw \quad (22) \]

As \( n \to +\infty \), we obtain that \( X \) follows the Poisson law of parameter \( \lambda = \lim_{n \to +\infty} \sum_{i=1}^{n} \lambda_i dw = \int_{0}^{\infty} \lambda(w) dw \). The remainder of the proof is standard. Following Mortensen (1986),

\[ \frac{\Pr (X = k)}{dt} = e^{-\lambda dt} \frac{\lambda^k (dt)^{k-1}}{k!} \quad \text{(23)} \]

The right-hand side tends to 0 when \( dt \) tends to 0 for all \( k > 1 \). Similarly, it tends to \( \lambda \) when \( dt \) tends to 0 when \( k = 1 \). It follows that the probability of receiving more than one offer conditional to the fact that the worker receives at least one offer tends to 0 when \( dt \) tends to 0.

4. Numerical simulations

Let \( m(\theta) = M_0 \theta^{-\alpha}, \ M_0 > 0, \ \alpha \in (0, 1), \ x(s) = s^{\gamma}, \ \gamma > 0, \ \text{and} \ c(S) = c_0 S^\sigma, \ \sigma \geq 1. \) Proposition 5 in the paper gives the normalized wage distribution, of which the actual wage distribution is a simple transformation. Indeed,

\[ G(x) = \Pr [w \leq x] \quad (24) \]

As \( \omega = (w - R) / (y - R) \), we get \( w = R + (y - R) \omega \). Therefore,

\[ G(x) = \Pr \left[ \omega \leq \frac{x - R}{y - R} \right] = H \left( \frac{x - R}{y - R} \right) \quad (25) \]

and

\[ G'(w) = \frac{1}{y - R} H' \left( \frac{w - R}{y - R} \right) \quad \text{(E0)} \]

where

\[ H' (\omega) = \frac{(1 - \omega)^{\frac{1 - \alpha}{\alpha} (\gamma + 1)} \omega^\gamma}{B \left( \frac{1 - \alpha}{\alpha} (\gamma + 1) + 1, \gamma + 1 \right) \ \forall \omega \in [0, 1]} \]

and \( B \) is the Beta function such that

\[ B (t_1 + 1, t_2 + 1) = \int_{0}^{1} (1 - \xi)^{t_1} \xi^{t_2} d\xi \quad (26) \]

The purpose of the numerical experiments is to depict \( G' \) as a function of various parameters. Changes in \( \alpha \) and \( \gamma \) not only affect the shape of \( H' \), they also modify the reservation wage \( R \) and the associated support of the wage distribution.
To find the reservation wage, we have to solve the following system of equations:

\[ \frac{k}{m[\theta(w)]} = \frac{y-w}{r+q}, \forall w \in [R, y] \] (27)

\[ c'(S) = x'[s(w)] \theta(w) m[\theta(w)] \frac{w-R}{r+q}, \forall w \in [R, y] \] (28)

\[ R = z - c(S) + \int_{R}^{y} x[s(w)] \theta(w) m[\theta(w)] \frac{w-R}{r+q} dw \] (29)

\[ S = \int_{R}^{y} s(w) dw \] (30)

**Lemma** The equilibrium overall search investment \( S^* \) and reservation wage \( R^* \) solve:

\[ (\sigma c_0)^{1+\gamma} S^{1+(\sigma-1)(1+\gamma)} = A I_1 \left( y-z - \left( \frac{1+\gamma}{\gamma} - 1 \right) c_0 S^\sigma \right)^{\frac{1+\gamma}{\alpha}} \] (E1)

\[ R = z + \left( \frac{1+\gamma}{\gamma} - 1 \right) c_0 S^\sigma \] (E2)

with

\[ A = \left[ \frac{\gamma}{1+\gamma} M_0^{1/\alpha} k^{1-1/\alpha} (r+q)^{-1/\alpha} \right]^{1+\gamma} \] (31)

\[ I_1 = \int_{0}^{1} (1-x)^{\frac{1-\alpha}{\alpha}} (1+\gamma) x^{1+\gamma} dx \] (32)

\[ = B \left( \frac{1-\alpha}{\alpha} (\gamma+1) + 1, 1+\gamma+1 \right) \] (33)

Proof. Equation (27) gives

\[ \theta(w) = \left( \frac{M_0}{k} \right)^{1/\alpha} \left( \frac{y-w}{r+q} \right)^{1/\alpha} \] (34)

\[ \theta(w) m(\theta(w)) = M_0^{1/\alpha} k^{1-1/\alpha} \left( \frac{y-w}{r+q} \right)^{1/\alpha-1} \] (35)

Now, equation (28) writes

\[ \frac{1+\gamma}{\gamma} s(w) c'(S) = x'[s(w)] \theta(w) m[\theta(w)] \frac{w-R}{r+q} \] (36)

We report the latter equation in equation (29) so as to obtain

\[ R = z - c(S) + \int_{R}^{y} \frac{1+\gamma}{\gamma} s(w) c'(S) dw \] (37)

\[ = z + \left( \frac{1+\gamma}{\gamma} - 1 \right) c(S) \] (38)
and (E2) is proved. Combining (35) and (36), we get
\[ s(w) c'(S)^{1+\gamma} = A (y - w)^{\frac{1}{\alpha} (1+\gamma)} (w - R)^{1+\gamma} \]  (39)

Therefore:
\[ Sc'(S)^{1+\gamma} = A \int_{R}^{y} (y - w)^{\frac{1}{\alpha} (1+\gamma)} (w - R)^{1+\gamma} dw \]  (40)
\[ = A I_1 (y - R)^{1+\gamma} + 1 \]  (41)

Combining (41) and (E2), we obtain (E1).

The Lemma allows us to proceed to the numerical simulations. The algorithm is the following:

Step 1. Fix the various parameters, compute A and solve the integral \( I_1 \).

Step 2. Find the root \( S^* \) of equation (E1) and compute \( R^* \) using equation (E2).

Step 3. Compute and plot the resulting density of the wage distribution (E0).

Step 4. Change the parameters and go to Step 1.

In addition to the wage distributions, we report the unemployment rate:
\[ u = \frac{q}{q + \lambda} \]  (42)
with
\[ \lambda = \int_{R}^{y} x [s(w)] \theta(w) m[\theta(w)] dw \]  (43)

Using (36), we get
\[ \lambda = (r + q) \frac{1 + \gamma}{\gamma} c'(S)^{\gamma} \int_{R}^{y} s(w) c'(S)^{1+\gamma} dw \]  (44)

Using (39), we obtain
\[ \lambda = A (r + q) \frac{1 + \gamma}{\gamma} c'(S)^{-\gamma} \int_{R}^{y} (y - w)^{\frac{1}{\alpha} (1+\gamma)} (w - R)^{\gamma} dw \]  (45)
\[ = A (r + q) \frac{1 + \gamma}{\gamma} c'(S)^{-\gamma} (y - R)^{1+\gamma} I_2 \]  (46)

where
\[ I_2 = \int_{0}^{1} (1 - \omega)^{\frac{1}{\alpha} (1+\gamma)} \omega^{\gamma} d\omega \]  (47)
\[ = B \left( \frac{1 - \alpha}{\alpha} (\gamma + 1) + 1, \gamma + 1 \right) \]  (48)

The value can be computed numerically once \( S^* \) and \( R^* \) have been found.